Spatial Separation of Events in S-Matrix Theory*

M. FROISSART
Centre d'Études Nucléaires de Saclay, Gif-sur-Yvette, Seine et Oise, France,

M. L. GOLDBERGER†
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey,

AND

K. M. WATSON
Department of Physics and Lawrence Radiation Laboratory, University of California, Berkeley, California

(Received 8 May 1963)

Just as the derivative of the argument of the S matrix with respect to energy gives a time interval for events, it is shown that the corresponding derivative with respect to momentum transfer gives a space interval. This space interval corresponds to the classical impact parameter in the classical limit. More generally, it is suggested that these two derivatives may provide a basis for introducing space-time intervals into physical theory.

I. INTRODUCTION

DURING the past few years there has been considerable interest in the possibility of replacing the ordinary dynamical description of physical systems via a Schrödinger equation by an S-matrix theory. The principal objection to the conventional theory is that it tells one rather more than he wants to know about a physical system; more precisely, it forces discussion of things that do not seem observable. One aspect of this problem that has been recently discussed is the notion of time interval in an S-matrix theory.[1] The idea was proposed that the S matrix, although superficially involving only information about the state of a system over long time intervals, does, in fact, provide a kind of coarse-grained definition of time interval. In a complex process, involving a sequence of operations, one can define a sequence of time intervals only to the extent that the S matrix for the entire event factors into a product of S matrices. When this is possible, a time label can be defined that involves only S-matrix (i.e., on energy shell) quantities. A dynamical principle may then be formulated from the S matrix for describing the change with time of physical systems.

It is natural to ask whether any analogous considerations apply for the definition of the spatial separation of events in an S-matrix theory. Such a description would evidently be "coarse grained," as was that for time intervals, and much more restrictive than the notion of a space-time continuum inherent in conventional field theory.

We shall see that a spatial separation for two interacting particles may indeed be defined in terms of the partial derivative of S with respect to the scattering angle. This quantity reduces to the classical impact parameter in the limit that a classical trajectory may be defined, and in general provides a *definition* for the impact parameter. In a manner analogous to that used for defining the time interval for a sequence of events, this impact parameter provides a means of constructing a trajectory for a particle undergoing a sequence of scatterings.

These, and the earlier considerations of time interval, suggest that a complete but coarse-grained description of space and time intervals may be derived in S-matrix theory, rather than postulated—as in conventional field theory.

II. WAVE-PACKET DESCRIPTION OF THE SCATTERING

For simplicity of discussion we restrict ourselves to the scattering of a simple spinless particle by a massive scatterer located at the origin of a given coordinate system. More complicated and physically interesting interactions would seem to involve complication of detail rather than of principle. The interaction and its observation involve directing a wave packet toward the scatterer at some initial time $t=-T_0$ and observing it at some later time T, as is illustrated in Fig. 1. We suppose that at both times $(-T_0)$ and T the wave packet is far from the scatterer. In the spirit of S-matrix theory we can assume that we know the wave function for the particle only at such times that it is far from the scatterer.

The wave function of the incident particle prior to interaction will be of the form

$$\phi(\mathbf{x},t)=(2\pi)^{-3/2}\exp[i(\mathbf{p}\cdot\mathbf{x}-\epsilon_p t)]G(\mathbf{x}-\mathbf{v}_0 t), \quad (1)$$

where $\mathbf{p}$, $\mathbf{v}_0$, and ϵ_p are, respectively, the initial momentum, velocity, and energy of the particle. The wave-packet amplitude

$$G(\mathbf{x})\equiv G(x,y,z) \quad (2)$$

is so constructed that at $t=0$ the packet is centered on

* This work was prepared under the auspices of the U. S. Atomic Energy Commission and a contract from the U. S. Air Force.
† Presently on leave at the Massachusetts Institute of Technology.
[1] M. L. Goldberger and K. M. Watson, Phys. Rev. **127**, 2284 (1962).

the scatterer at $\mathbf{x}=0$. More precisely, we write

$$\int d^3x\, \mathbf{x}\,|G(\mathbf{x})|^2=0. \tag{3}$$

The envelope G is assumed to have a spatial extent characterized by a length W. It is assumed to be "reasonably smooth" in the sense that its Fourier transform $a(\mathbf{l})$, in

$$G(x)=\int d^3l\, e^{i\mathbf{l}\cdot\mathbf{x}}a(\mathbf{l}), \tag{4}$$

is characterized by a "width" W^{-1} in momentum space. The width W is conveniently chosen large enough that spreading of the wave packet is negligible by the time it reaches the detector.[2] We shall also assume that over the momentum interval W^{-1} the S matrix, energy, and scattering amplitude are very nearly constant.

The wave function (1) may, with the assumptions just made, be written as

$$\phi(\mathbf{x},t)=(2\pi)^{-3/2}\int d^3\kappa\, \exp[i(\boldsymbol{\kappa}\cdot\mathbf{x}-\epsilon_\kappa t)]a(\boldsymbol{\kappa}-\mathbf{p}). \tag{5}$$

The momentum $\mathbf{p}$ is taken to be the mean momentum of the incident packet:

$$\mathbf{p}=\int d^3\kappa\boldsymbol{\kappa}\,|a(\boldsymbol{\kappa}-\mathbf{p})|^2. \tag{6}$$

The complete wave function for the scattering event is then

$$\psi(\mathbf{x},t)=\int d^3\kappa\psi_\kappa^+(\mathbf{x})e^{-i\epsilon_\kappa t}a(\boldsymbol{\kappa}-\mathbf{p}). \tag{7}$$

Here ψ_κ^+ is the steady-state wave function having the asymptotic form

$$\psi_\kappa^+(\mathbf{x})=(2\pi)^{-3/2}[\exp(i\boldsymbol{\kappa}\cdot\mathbf{x})+(e^{i\kappa x}/x)f(\kappa,\hat{x}\cdot\hat{\kappa})], \tag{8}$$

as $x\rightarrow\infty$. The quantity $f(\kappa,\hat{x}\cdot\hat{\kappa})$ is the amplitude for scattering from the initial direction $\hat{\kappa}$ to a final direction $\hat{x}$. The relation of f to the S matrix is described by the equations

$$S_{\kappa'\kappa}=\delta(\boldsymbol{\kappa}'-\boldsymbol{\kappa})-2\pi i\delta(\epsilon_{\kappa'}-\epsilon_\kappa)T_{\kappa'\kappa} \tag{9}$$

and

$$T_{\kappa'\kappa}=-[\kappa/(2\pi)^2\rho_\epsilon]f(\kappa,\hat{\kappa}'\cdot\hat{\kappa}), \tag{10}$$

where

$$\rho_\epsilon=\kappa^2/(d\epsilon_\kappa/d\kappa). \tag{11}$$

The separation $\mathbf{X}$ between the packet and scatterer is certainly observable, to within an accuracy of order W

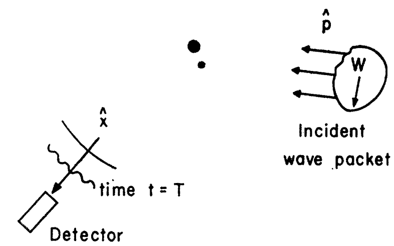

FIG. 1. Illustration of wave-packet scattering.

so long as $X\gg W$. The lower limit obtainable on W is determined from the properties of the interaction and the requirement of negligible spreading. The question that concerns us is whether the asymptotic wave functions alone (or the S matrix) permit one to describe the spatial separation of the particle and scattering center during the interaction.

To investigate this, we first use Eqs. (7) and (8) to write the asymptotic scattered wave as

$$\psi_{sc}(\mathbf{x},t)=(2\pi)^{-3/2}\int d^3\kappa\frac{\exp[i(\kappa x-\epsilon_\kappa t)]}{x}f(\kappa,\hat{x}\cdot\hat{\kappa})a(\boldsymbol{\kappa}-\mathbf{p}). \tag{12}$$

The complex scattering amplitude may evidently be written in the form

$$f(\kappa,\hat{x}\cdot\hat{\kappa})=R(\kappa,\hat{x}\cdot\hat{\kappa})\exp[i\chi(\kappa,\hat{x}\cdot\hat{\kappa})], \tag{13}$$

where R and χ are real.

Now, by our assumption that f varies little over the momentum interval W^{-1}, we may take

$$\epsilon_\kappa=\epsilon_p+\mathbf{l}\cdot\nabla_p\epsilon_p,\quad \kappa=p+\mathbf{l}\cdot\hat{p}, \tag{14}$$

and

$$f(\kappa,\hat{x}\cdot\hat{\kappa})=f(p,\hat{x}\cdot\hat{p})\exp(\mathbf{l}\cdot\nabla_p\ln R_0)\exp(i\mathbf{l}\cdot\nabla_p\chi_0) \tag{15}$$

in the integrand in (12). Here

$$\mathbf{l}=\boldsymbol{\kappa}-\mathbf{p}$$

and

$$R_0\equiv R(p,\hat{x}\cdot\hat{p}),\quad \chi_0=\chi(p,\hat{x}\cdot\hat{p}). \tag{16}$$

The factor $\exp(\mathbf{l}\cdot\nabla_p\ln R_0)$ in (15) leads to a distortion in the *shape* of the scattered wave packet. This is not of interest to us now, so we suppose it to be absorbed into the definition of the amplitude function a in Eq. (12). The second factor, $\exp(i\mathbf{l}\cdot\nabla_p\chi_0)$, leads to a *displacement* of the packet and does concern us. Indeed, on inserting the expressions (14) and (15) into (12), we find

$$\psi_{sc}(\mathbf{x},t)=(2\pi)^{-3/2}\frac{\exp[i(px-\epsilon_p t)]}{x}f(p,\hat{x}\cdot\hat{p})\times G[\hat{p}(x-v_0t)+\nabla_p\chi_0]. \tag{17}$$

[2] See, for example, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1963), Chap. III. When the wave packet has traveled a distance L to the detector, its amplitude will have been distorted to the form

$$G'=G\left[1+O\left(\frac{L}{\hbar v_0}\left(\frac{\hbar}{W}\right)^2\frac{1}{2M}\right)\right],$$

where M is the mass of the particle.

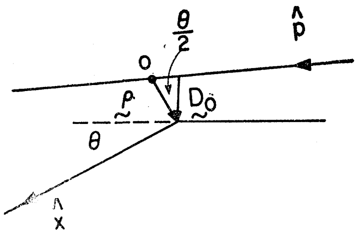

FIG. 2. Illustration of the vectors $\boldsymbol{\rho}$ and $\mathbf{D}_0$.

Here $\mathbf{v}_0=\hat{p}v_0=\boldsymbol{\nabla}_p\epsilon_p$ is the velocity of the incident particle.

For the validity of Eq. (17) we require that

$$\nabla_p{}^2\epsilon_p(T+T_0)\ll W^2,$$
$$\nabla_p{}^2\chi_0\ll W^3,$$

which are conditions placed on the wave packet.[2]

To give Eq. (17) a physical interpretation, we introduce

$$u\equiv\hat{x}\cdot\hat{p}$$

as a variable, and write

$$\boldsymbol{\nabla}_p\chi_0=\mathbf{v}_0(\partial\chi_0/\partial\epsilon_p)+(\hat{x}-\hat{x}\cdot\hat{p}\hat{p})(\partial\chi_0/p\partial u). \quad (18)$$

It is natural to call $\partial\chi_0/\partial\epsilon_p$ a "time delay,"

$$\tau_d\equiv(\partial\chi_0/\partial\epsilon_p)=(\partial\arg f/\partial\epsilon_p), \quad (19)$$

and to call

$$\mathbf{D}_0\equiv(\hat{x}-\hat{x}\cdot\hat{p}\hat{p})(\partial\chi_0/p\partial u) \quad (20)$$
$$=(\hat{x}-\hat{x}\cdot\hat{p}\hat{p})(\partial\arg f/p\partial u)$$

a "space shift." It may be noted that $\mathbf{D}_0$ is *perpendicular* to the incident direction $\mathbf{p}$.

The wave-packet amplitude in Eq. (17) has then the form

$$G=G\{\hat{p}[x-v_0(t-\tau_d)]+\mathbf{D}_0\}. \quad (21)$$

If, for example, the scattering lies in the $x-z$ plane of a rectangular coordinate system, with $\hat{p}$ directed along the z axis, we may write this in the notation of Eq. (2) as

$$G=G[D_0,\,0,\,x-v_0(t-\tau_d)]. \quad (22)$$

Equation (22) has a direct physical interpretation. Particles scattered into the direction $\hat{x}$ tend to be displaced off the z axis by a distance $\mathbf{D}_0$. This is illustrated in Fig. 2, where a "classical" trajectory is drawn. The displacement $\mathbf{D}_0$ is seen in this case to correspond to the classical impact parameter.

These considerations permit us to give a strictly quantum-mechanical definition of the *impact parameter* for a collision. In addition, we can define a *distance of closest approach* as the vector

$$\boldsymbol{\rho}=\rho[\hat{\imath}\cos(\theta/2)-\hat{p}\sin(\theta/2)], \quad (23)$$

where $\hat{\imath}$ is a unit vector parallel to the x axis, θ is the scattering angle ($\cos\theta=\hat{x}\cdot\hat{p}$), and

$$\rho=2\sin(\theta/2)(\partial\chi_0/p\partial u) \quad (24)$$
$$=[2\sin(\theta/2)/p](\partial\arg f/\partial u).$$

The expression (19) is a direct generalization of the Wigner–Eisenbud[3] time delay for scattering in pure eigenstates of the S matrix. The quantity τ_d evidently corresponds to a delay in the arrival of the packet at the detector. Its significance for the present considerations was discussed in Ref. 1.

We see from Eqs. (21) and (22) that a meaning can be given to the term "spatial separation" of two interacting particles. In the next section we give a different, and more direct, calculation of this quantity.

III. DIRECT CALCULATION OF POSITION OF THE WAVE PACKET

We discuss once again the same scattering event that was described in Sec. II, but now calculate directly the center of mass of the packet.

If there were no scattering, the wave function (5) would describe the packet motion. Its mean initial position at the time $t=-T_0$ is then

$$\mathbf{X}_0{}^0=\int ed^3x\,\mathbf{x}|\phi(\mathbf{x},\,-T_0)|^2. \quad (25)$$

In the absence of scattering, and at the time T, the mean position of the packet is

$$\mathbf{X}_0(T)=\int d^3x\,\mathbf{x}|\phi(\mathbf{x},T)|^2. \quad (26)$$

When scattering occurs we must use the wave function (7) to find the packet location. At the time $T=-T_0$ this is

$$\mathbf{X}_0=\int d^3x\,\mathbf{x}|\psi(\mathbf{x},\,-T_0)|^2. \quad (27)$$

Since $(-T_0)$ was chosen as a time long before scattering occurred, we will have

$$\psi(\mathbf{x},\,-T_0)=\phi(\mathbf{x},\,-T_0), \quad (28)$$

or

$$\mathbf{X}_0=\mathbf{X}_0{}^0.$$

To find the position of the scattered wave packet at time T for those particles scattered into the direction $\hat{x}$, we introduce a projection operator $\Lambda(\hat{x})$ onto those plane-wave states corresponding to momentum vectors parallel to $\hat{x}$ and lying in the small increment $\delta\Omega(\hat{x})$ of solid angle. The required mean coordinate of the wave packet is then

$$\mathbf{X}(T)=\frac{1}{N}\int d^3x\,\mathbf{x}|\Lambda(\hat{x})\psi(\mathbf{x},T)|^2, \quad (29)$$

where

$$N\equiv\int d^3x|\Lambda(\hat{x})\psi(\mathbf{x},T)|^2. \quad (30)$$

[3] E. P. Wigner, Phys. Rev. **98**, 145 (1955), and L. Eisenbud, Ph.D. thesis, Princeton University, 1948 (unpublished).

The displacement due to scattering is

$$\Delta\mathbf{X}\equiv[\mathbf{X}(T)-\mathbf{X}_0]-[\mathbf{X}^0(T)-\mathbf{X}_0{}^0]$$
$$=\mathbf{X}(T)-\mathbf{X}^0(T). \quad (31)$$

It will be convenient and will involve no serious loss of generality to suppose that $\hat{x}$ is so directed that the waves scattered to the detector do not overlap the nonscattered waves in the incident packet.

The quantities (27) and (29) do not, of course, exhaust the averages that may be evaluated for a description of the "particles orbit." For example, higher moments may also be found by the method described here.

To calculate the expressions (25) and (26), we shall use Eq. (5) for $\phi(\mathbf{x},t)$. For the expression (29) we shall find it convenient to write $\psi(\mathbf{x},T)$ in the form[4]

$$\psi(\mathbf{x},T)=(2\pi)^{-3/2}\int d^3\kappa'\int d^3\kappa \times\exp[i(\boldsymbol{\kappa}'\cdot\mathbf{x}-\epsilon_{\kappa'}T)]S_{\kappa'\kappa}a(\boldsymbol{\kappa}-\mathbf{p}), \quad (32)$$

where $S_{\kappa'\kappa}$ is the S-matrix element (9).

Let us first evaluate $\mathbf{X}^0(T)$, Eq. (26). Using Eq. (5), we obtain

$$\mathbf{X}^0(T)=(2\pi)^{-3}\int d^3x\,\mathbf{x}\int d^3\kappa_1 d^3\kappa_2 \times\exp\{i[(\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_1)\cdot\mathbf{x}-(\epsilon_{\kappa_2}-\epsilon_{\kappa_1})T]\} \times a^*(\boldsymbol{\kappa}_1-\mathbf{p})a(\boldsymbol{\kappa}_2-\mathbf{p}). \quad (33)$$

If we define

$$\boldsymbol{\kappa}_2\equiv\mathbf{K}+\tfrac{1}{2}\mathbf{l} \quad (34)$$

and

$$\boldsymbol{\kappa}_1\equiv\mathbf{K}-\tfrac{1}{2}\mathbf{l},$$

we may write

$$\mathbf{x}=(1/i)\boldsymbol{\nabla}_l \quad (35)$$

in the integrand of Eq. (33). Then, after performing a partial integration, we find

$$\mathbf{X}^0(T)=\int d^3x\int d^3\kappa_1 d^3\kappa_2\{(2\pi)^{-3}\exp[i(\boldsymbol{\kappa}_2-\boldsymbol{\kappa}_1)\cdot\mathbf{x}]\} \times\{-(1/i)\boldsymbol{\nabla}[\exp(-i\mathbf{l}\cdot\nabla_K\epsilon_K T)] \times[a^*(\mathbf{K}-\tfrac{1}{2}\mathbf{l}-\mathbf{p})a(\mathbf{K}+\tfrac{1}{2}\mathbf{l}-\mathbf{p})]\}$$
$$=\int d^3\kappa\Big[|a(\boldsymbol{\kappa}-\mathbf{p})|^2T(\nabla_\kappa\epsilon_\kappa) +\frac{i}{2}a^*(\boldsymbol{\kappa}-\mathbf{p})\overleftrightarrow{\nabla}_\kappa a(\boldsymbol{\kappa}-\mathbf{p})\Big]. \quad (36)$$

Here we have used the notation

$$f\overleftrightarrow{\nabla}g\equiv f(\nabla g)-g\nabla f. \quad (37)$$

[4] The scattering is long since past at the time T, we recall. (See, for example, Ref. 2, Chap. V.)

Since $\nabla_\kappa\epsilon_\kappa$ is the velocity of the particle when its momentum is $\boldsymbol{\kappa}$, we may use Eq. (6) to write the first term in Eq. (36) as

$$T\int d^3\kappa(\nabla_\kappa\epsilon_\kappa)|a(\boldsymbol{\kappa}-\mathbf{p})|^2=\mathbf{v}_0T, \quad (38)$$

where $\mathbf{v}_0$ is the incident velocity of the packet [see Eq. (1)]. On transforming the second term to coordinate space, we then have

$$\mathbf{X}^0(T)=\mathbf{v}_0T+\int d^3x\,\mathbf{x}|\phi(\mathbf{x},0)|^2 \quad (39)$$
$$=\mathbf{v}_0T,$$

because of the condition (3).

We next simplify Eq. (30) for N. Using Eq. (32), we may write this as

$$N=(2\pi)^{-3}\int d^3x\int_{\hat{x}} d^3\kappa_1'\int_{\hat{x}} d^3\kappa_2' \times\Big\{\int d^3\kappa_1\exp[i(\boldsymbol{\kappa}_1'\cdot\mathbf{x}-\epsilon_{\kappa_1}'T)]S_{\kappa_1'\kappa_1}a(\boldsymbol{\kappa}_1-\mathbf{p})\Big\}^* \times\Big\{\int d^3\kappa_2\exp[i(\boldsymbol{\kappa}_2'\cdot\mathbf{x}-\epsilon_{\kappa_2}'T)]S_{\kappa_2'\kappa_2}a(\boldsymbol{\kappa}_2-\mathbf{p})\Big\}$$
$$=\int_{\hat{x}} d^3\kappa'\int d^3\kappa_1 d^3\kappa_2 a^*(\boldsymbol{\kappa}_1-\mathbf{p})S^*_{\kappa'\kappa_1}S_{\kappa'\kappa_2}a(\boldsymbol{\kappa}_2-\mathbf{p}). \quad (40)$$

Here $\int_{\hat{x}}d^3\kappa'\cdots$ denotes an integral over $\boldsymbol{\kappa}'$ with the direction $\boldsymbol{\kappa}'$ restricted to the solid angle $\delta\Omega(\hat{x})$, as implied by the projection operator $\Lambda(\hat{x})$ in Eqs. (29) and (30). Because we have assumed that waves scattered in the direction $\hat{x}$, do not overlap the incident packet at time T [this implies that $a(\kappa\hat{x}-\mathbf{p})$ is negligibly small], the δ-function terms do not contribute, and N reduces to

$$N=(2\pi)^2\int d^3\kappa'\int d^3\kappa_1 d^3\kappa_2\delta(\epsilon_{\kappa'}-\epsilon_{\kappa_1})\delta(\epsilon_{\kappa'}-\epsilon_{\kappa_2})T^*_{\kappa'\kappa_1}T_{\kappa'\kappa_2} \times a^*(\boldsymbol{\kappa}_1-\mathbf{p})a(\boldsymbol{\kappa}_2-\mathbf{p}). \quad (41)$$

The assumption that $T_{\kappa'\kappa_2}$ and $T^*_{\kappa'\kappa_1}$ are essentially constant over the packet now permits us to write this as

$$N=(2\pi)^2\rho_\epsilon\delta\Omega(\hat{x})|T_{\mathbf{kp}}|^2 \times\int d^3\kappa_1 d^3\kappa_2\delta(\epsilon_{\kappa_1}-\epsilon_{\kappa_2})a^*(\boldsymbol{\kappa}_1-\mathbf{p})a(\boldsymbol{\kappa}_2-\mathbf{p}), \quad (42)$$

where

$$\mathbf{k}=p\hat{x} \quad (43)$$

is the momentum of the scattered particle and ρ_ϵ is the expression (11), now evaluated at $\kappa=p$.

Let us next substitute

$$\delta(\epsilon_{\kappa 1}-\epsilon_{\kappa 2})=(2\pi)^{-1}\int_{-\infty}^{+\infty}dt\,\exp[i(\epsilon_{\kappa 1}-\epsilon_{\kappa 2})t] \quad (44)$$

into Eq. (42) and use Eq. (10) to obtain

$$N=\frac{k^2}{\rho_\epsilon}\delta\Omega(\hat{x})|f(p,\hat{x}\cdot\hat{p})|^2\int_{-\infty}^{+\infty}dt|\phi(0,t)|^2. \quad (45)$$

Now,

$$d\sigma(\hat{p}\rightarrow\hat{x})=|f(p,\hat{x}\cdot\hat{p})|^2\delta\Omega(\hat{x}) \quad (46)$$

is the differential scattering cross section, and

$$F_0=\frac{k^2}{\rho_\epsilon}\int_{-\infty}^{+\infty}dt|\phi(0,t)|^2 \quad (47)$$

is the flux of particles (per incident particle) on the scattering center. Thus, we finally have

$$N=d\sigma(\hat{p}\rightarrow\hat{x})F_0. \quad (48)$$

To be strictly consistent, we should have kept the first-order variation of T and T^* with momentum [as in Eq. (15)] in Eq. (42). This would have led to a flux (47) evaluated at the displaced position $\mathbf{D}_0$ [Eq. (20)]. Since this correction does not affect our results, we have avoided the algebraic complication of including it here.

We turn next to the evaluation of the quantity [see Eq. (29)]

$$\begin{aligned}N\mathbf{X}(T)&=\int d^3x\,\mathbf{x}|\Lambda(\hat{x})\psi(\mathbf{x},T)|^2\\&=\int d^3x\,\mathbf{x}\int_{\hat{x}}d^3\kappa_1'\int_{\hat{x}}d^3\kappa_2'\\&\times\left[\int d^3\kappa_1\exp i(\boldsymbol{\kappa}_1'\cdot\mathbf{x}-\epsilon_{\kappa_1}'T)S_{\kappa_1'\kappa_1}a(\boldsymbol{\kappa}_1-\mathbf{p})\right]^*\\&\times\left[\int d^3\kappa_2\exp i(\boldsymbol{\kappa}_2'\cdot\mathbf{x}-\epsilon_{\kappa_2}'T)S_{\kappa_2'\kappa_2}a(\boldsymbol{\kappa}_2-\mathbf{p})\right],\end{aligned} \quad (49)$$

which, of course, differs from the first form of Eq. (40) only by the factor $\mathbf{x}$ in the integrand. This expression may be simplified by introducing the variables (34) and using Eq. (35) for $\mathbf{x}$. The steps leading to Eq. (36) now give

$$\begin{aligned}N\mathbf{X}(T)=\int_x d^3\kappa'\int d^3\kappa_1 d^3\kappa_2\\&\times\{(\nabla_{\kappa'}\epsilon_{\kappa'})Ta^*(\boldsymbol{\kappa}_1-\mathbf{p})S^*_{\kappa'\kappa_1}S_{\kappa'\kappa_2}a(\boldsymbol{\kappa}_2-\mathbf{p})\\&+(i/2)[a^*(\boldsymbol{\kappa}_1-\mathbf{p})S^*_{\kappa'\kappa_1}]\overleftrightarrow{\nabla}_{\kappa'}[S_{\kappa'\kappa_2}a(\boldsymbol{\kappa}_2-\mathbf{p})]\}.\end{aligned} \quad (50)$$

Were we to set $S_{\kappa'\kappa_2}=\delta(\boldsymbol{\kappa}'-\boldsymbol{\kappa}_2)$ and $S^*_{\kappa'\kappa_1}=\delta(\boldsymbol{\kappa}'-\boldsymbol{\kappa}_1)$ and integrate over all $\boldsymbol{\kappa}'$, this would agree with Eq. (36)—as it should, because in the absence of scattering $\Delta\mathbf{X}=0$.

Since $\delta\Omega(\hat{x})$ is very small, we may factor $(\nabla_{\kappa'}\epsilon_{\kappa'})T=\mathbf{v}_fT$, where $\mathbf{v}_f=v_0\hat{x}$, out of the integrand of the first term in Eq. (50) and rewrite this as

$$\begin{aligned}N\mathbf{X}(T)&=\mathbf{v}_fTN\\&+\frac{i}{2}\int_{\hat{x}}d^3\kappa'\int d^3\kappa_1d^3\kappa_2[a^*(\boldsymbol{\kappa}_1-\mathbf{p})S^*_{\kappa'\kappa_1}]\\&\times\overleftrightarrow{\nabla}[a(\boldsymbol{\kappa}_2-\mathbf{p})S_{\kappa'\kappa_2}].\end{aligned} \quad (51)$$

[Compare Eq. (51) to the final form of Eq. (40).]

The leading term for large T in $\mathbf{X}(T)$ is just $\mathbf{v}_fT$, as would be expected from elementary kinematical considerations. The second term on the right in Eq. (51) is independent of T and corresponds to a displacement of the particle trajectory.

To simplify this term we substitute the expression (9) for the S matrix and again use the condition that $a(p\hat{x}-\mathbf{p})\approx 0$ to write

$$\begin{aligned}N[\mathbf{X}(T)-\mathbf{v}_fT]\\&=\frac{i}{2}(2\pi)^2\int_{\hat{x}}d^3\kappa'\int d^3\kappa_1d^3\kappa_2a^*(\boldsymbol{\kappa}_1-\mathbf{p})a(\boldsymbol{\kappa}_2-\mathbf{p})\\&\times\{[\delta(\epsilon_{\kappa'}-\epsilon_{\kappa_1})T^*_{\kappa'\kappa_1}]\overleftrightarrow{\nabla}_{\kappa'}[\delta(\epsilon_{\kappa'}-\epsilon_{\kappa_2})T_{\kappa'\kappa_2}]\}.\end{aligned} \quad (52)$$

Now,

$$\begin{aligned}\nabla_{\kappa'}[T_{\kappa'\kappa_2}\delta(\epsilon_{\kappa'}-\epsilon_{\kappa_2})]\\&=\delta(\epsilon_{\kappa'}-\epsilon_{\kappa_2})(\nabla_{\kappa'}T_{\kappa'\kappa_2})\\&+T_{\kappa'\kappa_2}\left(-\frac{\mathbf{v}'\mathbf{v}_2}{v_2^2}\cdot\nabla_{\kappa_2}\right)\delta(\epsilon_{\kappa'}-\epsilon_{\kappa_2}),\end{aligned}$$

etc., where $\mathbf{v}'\equiv\nabla_{\kappa'}\epsilon_{\kappa'}=\mathbf{v}_f$. This permits us to put Eq. (52) into the form

$$\begin{aligned}N[\mathbf{X}(T)-\mathbf{v}_fT]\\&=\frac{i}{2}(2\pi)^2\rho_\epsilon\delta\Omega(\hat{x})\int d^3\kappa_1d^3\kappa_2\delta(\epsilon_{\kappa_1}-\epsilon_{\kappa_2})\\&\times a^*(\boldsymbol{\kappa}_1-\mathbf{p})a(\boldsymbol{\kappa}_2-\mathbf{p})[T^*_{\kappa'\kappa}(\overleftrightarrow{\nabla}_{\kappa'}+\hat{x}\hat{p}\cdot\overleftrightarrow{\nabla}_{\kappa})T_{\kappa'\kappa}]|_{\kappa'=k,\kappa=p}\\&+\text{ terms that vanish for small }1/W.\end{aligned} \quad (53)$$

(The neglected terms here involve gradients of the a's and thus depend on wave-packet characteristics. When $1/W$ is small enough that we can set $\mathbf{v}_1\approx\mathbf{v}_2\approx\mathbf{v}_0$, these terms vanish.)

To further simplify Eq. (53), we use Eq. (10) and Eqs. (44) and (47) to obtain

$$N[\mathbf{X}(T)-\mathbf{v}_fT]=-N(\nabla_k+\hat{x}\hat{p}\cdot\nabla_p)\arg f(p,\hat{x}\cdot\hat{p}). \quad (54)$$

Now,

$$-(\nabla_k+\hat{x}\hat{p}\cdot\nabla_p)\arg f(p,\hat{x}\cdot\hat{p})=-\tau_d\mathbf{v}_f+\mathbf{D}_f, \quad (55)$$

where τ_d is the time delay (19) and

$$\mathbf{D}_f\equiv-(\hat{p}-\hat{x}\hat{x}\cdot\hat{p})(\partial\arg f/p\partial u), \quad (56)$$

and $u=\hat{x}\cdot\hat{p}$ as in Eq. (18). Finally, we use Eqs. (39) and (54) to write Eq. (30) as

$$\mathbf{X}(T)=\mathbf{v}_f(T-\tau_d)+\mathbf{D}_f, \quad (57)$$

or

$$\Delta\mathbf{X}=\mathbf{v}_f(T-\tau_d)-\mathbf{v}_0T+\mathbf{D}_f. \quad (58)$$

The interpretation of these expressions is similar to that given in Sec. II of Eq. (22). The scattered wave packet is delayed by a time τ_d and displaced a distance $\mathbf{D}_f$, which lies in the plane of the scattering and is perpendicular to the direction $\hat{x}$. This is illustrated in Fig. 3, where Eqs. (20) and (57) are used to define a "trajectory" for the particle.

Referring to Fig. 3, we see that if the scattering had "actually occurred" at O, the point $\mathbf{X}(T)$ would have been at P. Because the scattering is displaced by the distance $\boldsymbol{\rho}$ [see Eq. (23)], the point $\mathbf{X}(T)$ is displaced by a distance D_f perpendicular to the line OP. The displacement of the incident orbit is $\mathbf{D}_0$ [see Eq. (20)]. We see that

$$D_f=D_0=\rho\cos(\theta/2), \quad (59)$$

and D_f is in the direction of the unit vector $\hat{e}$, illustrated in Fig. 3.

Our discussion has been quite general to this point and certainly consistent with the indeterminacy principle. The "trajectory" drawn in Fig. 3 has been defined in terms of the *mean displacement* $\mathbf{D}_0$ and $\mathbf{D}_f$. In the next section we shall evaluate these quantities in the classical limit and see that $\boldsymbol{\rho}$ does indeed then correspond to just the classical distance of closest approach.

Before doing this, let us suppose that the scattering interaction illustrated in Fig. 3 is *weak* and *limited* to small angles θ, and that the orbit may be considered as classical. The displacement $\mathbf{D}_0$ and $\mathbf{D}_f$ are then directly interpretable as displacements of the classical trajectory from QOP. The time delay τ_d requires discussion, however. There are two contributions to τ_d. One results from the fact that the trajectory RSX is shorter than QOP by the line segments aO and bS. Since, this length is $2\rho\sin(\theta/2)$, we have a purely geometrical contribution to τ_d,

$$\tau_{\text{geom}}=-(2\rho/v_0)\sin(\theta/2). \quad (60)$$

The time delay also has a dynamical contribution corresponding to the fact that the velocity of the particle is in general different while it is interacting. To evaluate this in the classical limit, we suppose that the scattering is due to a potential $V(r,z)$, where z is a coordinate along $\hat{p}$, and r a coordinate along $\hat{j}$. Now, the velocity v, if the particle has a nonrelativistic

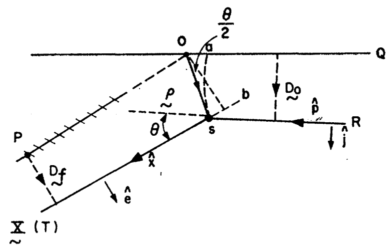

FIG. 3. Construction of "classical" trajectory.

energy, at (r,z) is given by the equation

$$v^2+(2/M)V(r,z)=v_0^2, \quad (61)$$

where M is the particle mass. Since we have assumed that θ is small and that $(2/M)|V|\ll v_0^2$, we obtain from Eq. (61) for an impact parameter ρ

$$dt\approx\frac{dz}{v_0}\left[1+\frac{1}{Mv_0^2}V(\rho,z)\right],$$

or

$$\tau_{\text{dyn}}=\frac{1}{pv_0^2}\int_{-\infty}^{+\infty}V(\rho,z)dz \quad (62)$$

for the *dynamical* contribution to the time delay. The total time delay τ_d is then

$$\tau_d=\tau_{\text{dyn}}+\tau_{\text{geom}}. \quad (63)$$

IV. SCATTERING IN THE NEAR-CLASSICAL LIMIT

Let us evaluate the scattering illustrated in Fig. 3 in the WKBJ, or eikonal, approximation for the case of a nonrelativistic particle. Then, if the scattering is due to a potential $V(r,z)$ and is limited to small angles θ,[5] the scattering amplitude[6] is

$$f(p,\hat{x}\cdot\hat{p})=-ip\int_0^\infty rdr\,J_0(pr\theta)[e^{2i\delta(r,p)}-1], \quad (64)$$

where J_0 is the Bessel function of zero order and

$$\delta(r,p)=-\frac{1}{2v_0}\int_{-\infty}^{+\infty}dzV(r,z). \quad (65)$$

In the near-classical limit we may replace J_0 by its asymptotic form to write

$$f\approx-i\left(\frac{p}{2\pi\theta}\right)^{1/2}\int_0^\infty r^{1/2}dr \times\{\exp[i(pr\theta-\tfrac{1}{4}\pi)]+\exp[-i(pr\theta-\tfrac{1}{4}\pi)]\} \times(e^{2i\delta}-1), \quad (66)$$

which may easily be evaluated by a saddle-point integration. To do this, we must consider the two

[5] The limitation to small scattering angles is not essential here, but simplifies our discussion.
[6] See Ref. 2, Eq. (6-505), for example.

integrals

$$I^{\pm}=\int_0^{\infty} r^{1/2}dr e^{i\phi^{\pm}}, \tag{67}$$

where

$$\phi^{\pm}{}_{(r,p)}=2\delta(r,p)\pm(pr\theta-\tfrac{1}{4}). \tag{68}$$

The stationary phase point at $r=\rho_0$ is determined from the equations

$$2\delta'(\rho_0,p)\pm p\theta=0, \tag{69}$$

where $\delta'\equiv\partial\delta(r,p)/\partial r$.

Now,

$$\delta'(r,p)=\frac{1}{2v_0}\int_{-\infty}^{+\infty}dz\left[-\frac{\partial V(r,z)}{\partial r}\right], \tag{70}$$

so

$$\delta'(r,p)>0 \text{ for a repulsive force (case R)},$$
$$\delta'(r,p)<0 \text{ for an attractive force (case A)}.$$

We see then that in the present approximation

$$f=-i(p/2\pi\theta)^{1/2}I^{\pm}, \tag{71}$$

where the plus sign corresponds to case A and the minus sign to case R. Evaluation of $I^{\pm}$ gives

$$f=-i(p\rho_0/|\theta_0''|)^{1/2}e^{\pm i\frac{1}{4}\pi}e^{i\phi_0^-}, \quad \text{case R}$$
$$f=-i(p\rho_0/|\phi_0''|)^{1/2}e^{\pm i\frac{1}{4}\pi}e^{i\phi_0^+}, \quad \text{case A}, \tag{72}$$

where $\phi_0^{\pm}\equiv\phi^{\pm}(\rho_0,p)$ and $\phi_0''\equiv\phi''(\rho_0,p)$. The plus sign in Eqs. (72) is to be used when $\phi_0''>1$, the minus sign for $\phi_0''<1$.

Using Eqs. (24) and (72), we find the impact parameter ρ to be

$$\begin{aligned}\rho&=\rho_0, &&\text{case R},\\ &=-\rho_0, &&\text{case A},\end{aligned} \tag{73}$$

in agreement with our anticipations.

The time delay (17) is evaluated from Eqs. (72) as

$$\begin{aligned}\tau_d&=\frac{1}{v_0}2\frac{\partial\delta}{\partial p}\pm\frac{\rho_0\theta}{v_0}\\ &=+\frac{1}{Mv_0^3}\int_{-\infty}^{+\infty}dzV(\rho_0,z)\pm\frac{\rho_0\theta}{v_0}\end{aligned} \tag{74}$$

by using Eq. (65). For case R and small θ (minus sign), this is seen to agree precisely with Eqs. (60), (62), and (63).

V. AN ALTERNATIVE REPRESENTATION

We have considered the scattering amplitude to be a function of ϵ_p and $u=\hat{x}\cdot\hat{p}$, and have shown that the derivatives (19) and (24) of arg f with respect to these variables have a simple geometrical interpretation. If one considers f to be a function of the variables[7]

$$\begin{aligned}s&\equiv 2M\epsilon_p,\\ t&\equiv -2s(1-u),\end{aligned} \tag{75}$$

rather than of ϵ_p and u, the partial derivatives of arg $f(s,t)$ may be given a dynamical interpretation.

To see this, let us first generalize the definition (62) for τ_{dyn}, writing

$$\tau_{\text{dyn}}\equiv\tau_d-\tau_{\text{geom}}, \tag{76}$$

where τ_d is defined by Eq. (19) and τ_{geom} by Eq. (60). An elementary calculation then gives

$$2(s)^{1/2}\frac{\partial\,\arg f(s,t)}{\partial t}=\frac{\rho}{2\sin(\theta/2)}, \tag{77}$$

where ρ is defined by Eq. (24), and

$$2M\frac{\partial\,\arg f(s,t)}{\partial s}\equiv\tau_c=\tau_{\text{dyn}}. \tag{78}$$

Here τ_{dyn} is defined by Eq. (76).

We call the quantity τ_c the "causal time delay." Equation (62) suggests that this has a more direct dynamical significance than does τ_d.

VI. CONSTRUCTION OF A TRAJECTORY

In Ref. 1 it was observed that for a sequence of scatterings, or in the quasi-classical limit, for which the S matrix factors into a product of S matrices, the time delay τ_d permits one to attach a coarse-grained time label to points on the trajectory. In a similar manner we can use Eqs. (23) and (24) to construct an "orbit" in coordinate space for the scattered particle. That is, when

$$S=\textstyle\prod_i S_i,$$

where S_i is an S matrix for the ith scattering, we may define a sequence $\boldsymbol{\varrho}_i$ of displacement parameters. A path formed by line segments between this sequence of vectors provides the required "orbit." It is evident that in the classical limit this orbit will coincide with the classical trajectory.

We have seen that the S matrix may provide a basis for defining space-time intervals for events. The extent to which it may provide a general and satisfactory definition of space-time intervals is not presently clear.

[7] A relativistic generalization is evidently straightforward.